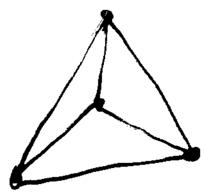
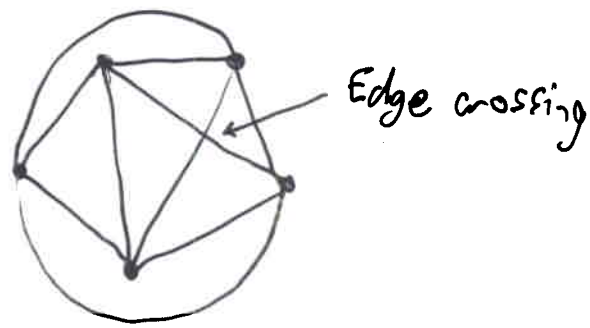
Planar Graphs

"Embeddable" in the plane



Yes



No

What is an embedding?

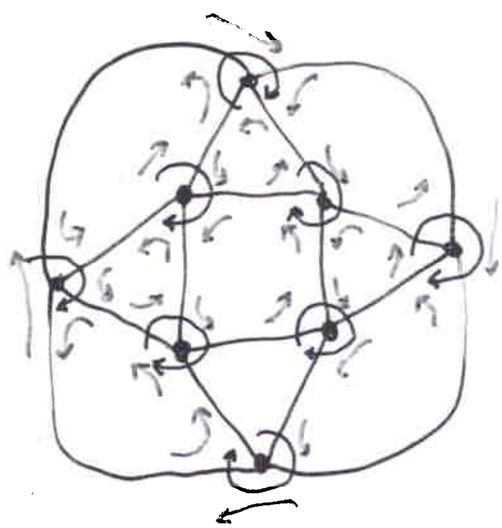
1. A (straight-line) drawing connecting (lattice) points

   (Yes, even integer coordinates $O(n)$)

2. A circular ordering of edges around each vertex, specifying faces, satisfying Euler's formula

   $$V + F = E + 2$$

   Face = a connected region bounded by vertices and edges



(n) $V = 8$

$F = 10$

(m) $E = 16$

$V + F = E + 2$ (connected planar graph; formula must be modified for # components)

Proof by induction

True for a tree $F = 1$

$E = V - 1$

$E + 2 = V + 1 = V + F$

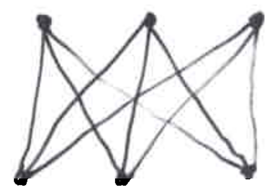
Start with a spanning tree

Adding one edge adds one to E and one to F, preserving the equality

Kuratowski's Theorem

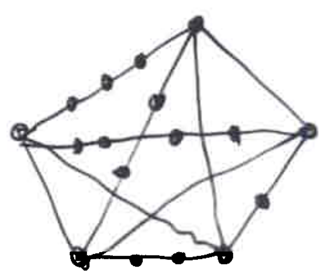
Planar iff no extended K_5 or $K_{3,3}$ as a subgraph

K_5



$K_{3,3}$

"extended": each edge is a path, vertex-disjoint except for ends (add vertices along edges)



extended K_5

Planar graphs are sparse:

$E \leq 3V - 6$ for $V \geq 3$ from Euler's formula

$\Rightarrow$ ave degree < 6

Planarity is preserved by edge deletion or edge contraction

Planar graphs are 5-colorable (easy)

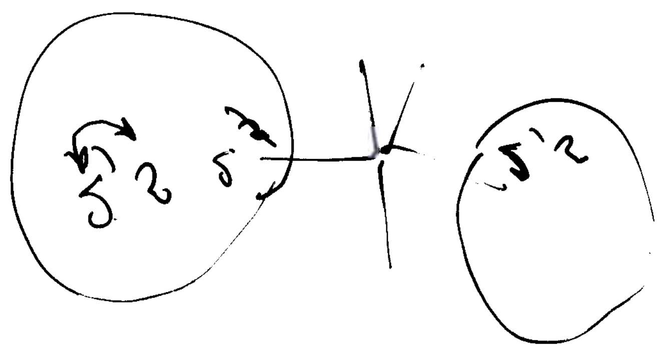
Planar graphs are 4-colorable (hard)

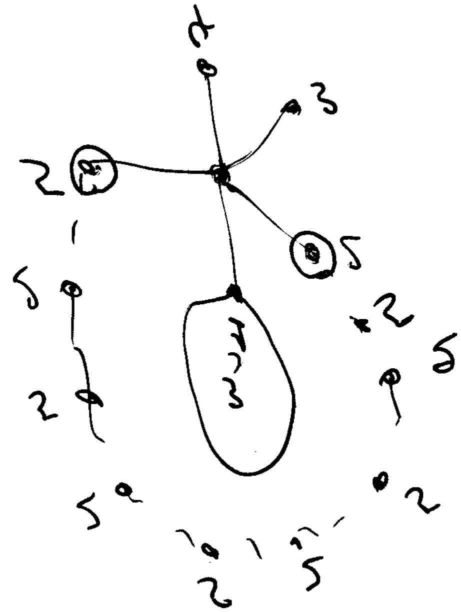
Planar graphs have small separators:

removal of $O(\sqrt{n})$ vertices leaves no connected component exceeding $(2/3)n$ vertices

separator can be found in $O(n)$ time

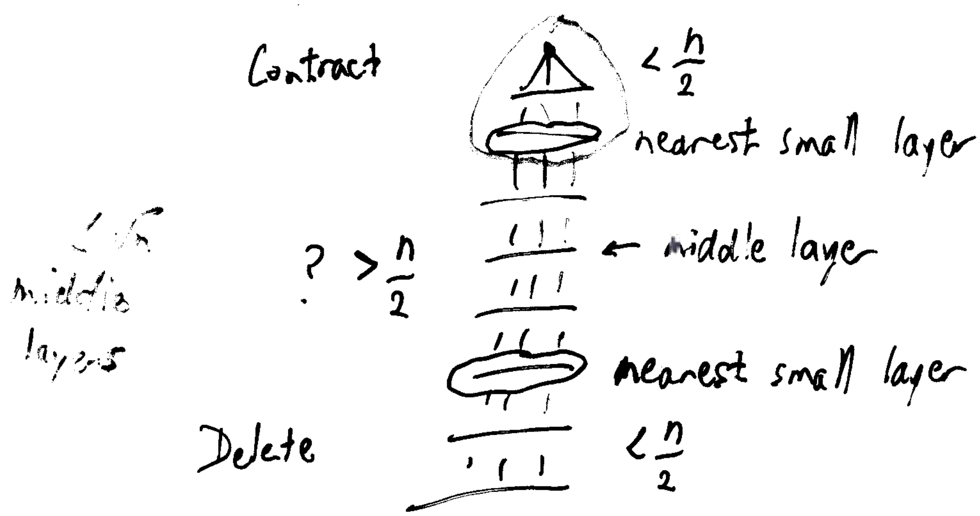
Planarity can be tested, and an embedding found, in $O(n)$ time

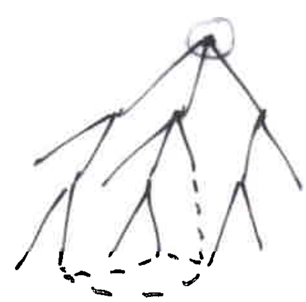




Proof of separator theorem

Breadth-first search





Shallow spanning tree

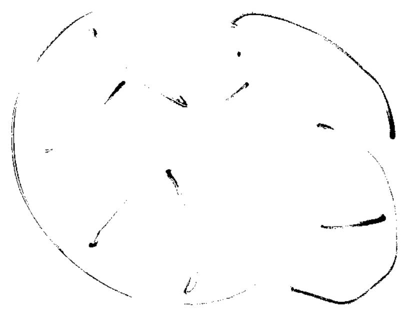
Each nontree edge defines a cycle.

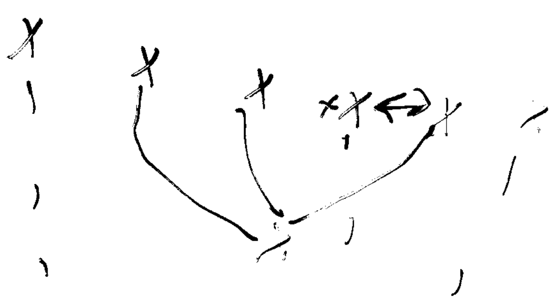
Such a cycle partitions graph between inside, outside. One such cycle gives a 2/3 - 1/3 or better split.

Two linear-time planarity algorithms

Via depth-first search, path decomposition,
tracking embedding possibilities via
stack of stacks

Via bipolar order, vertex-by-vertex, tracking
embedding possibilities via PQ tree





○ = P-nodes = arb permute

□ - Q-nodes = mirror image